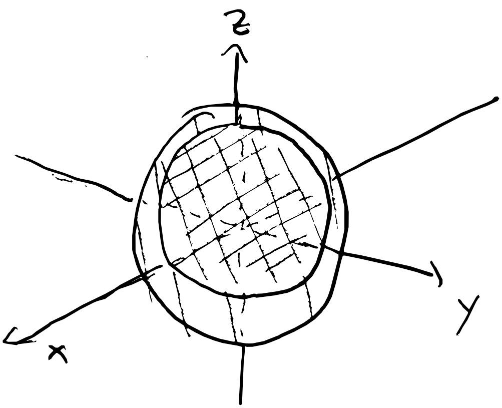
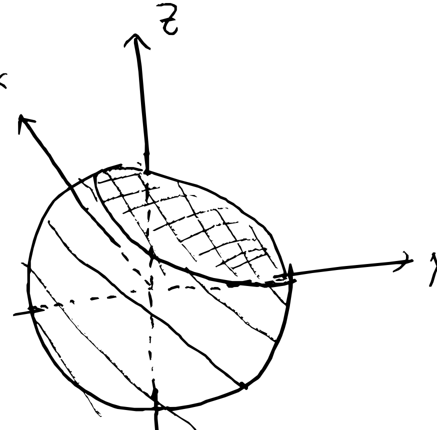
Last time we learned that we can use the method of <u>Lagrange multipliers</u> whenever there is an equality in the definition of the domain.

$\boxed{\text{Question}}$ What if there are more than one equality?

Such a case can also be treated with the Lagrange multipliers, just slightly more complicated.

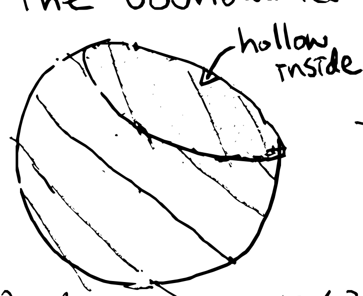
$\boxed{\text{Example}}$ Consider the domain $\{ x^2+y^2+z^2 \le 9,\ x+y+z \le 3 \}$





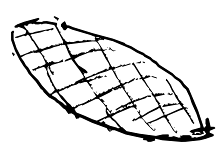
(other perspective)

The boundaries have two parts:



$x^2+y^2+z^2=9,\ x+y+z \le 3$

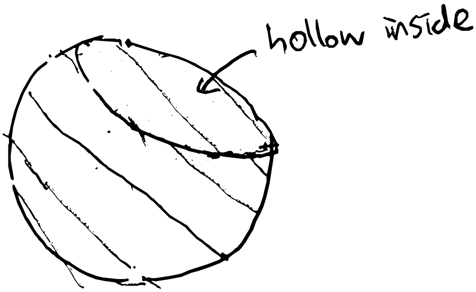
$+$



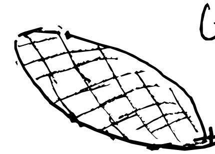
(this is a disk)

$x^2+y^2+z^2 \le 9,\ x+y+z=3$

These boundaries have a common boundary:

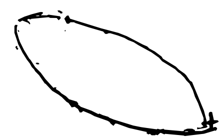


$x^2+y^2+z^2=9, \ x+y+z\le 3$



(this is a disk)

$x^2+y^2+z^2\le 9, \ x+y+z=3$



(this is a ring)

$x^2+y^2+z^2=9, \ x+y+z=3$

This "boundary of the boundaries", which must appear in Step 3 of Step 3 of Step 3, is defined using 2 equalities.

Q. How do we systematically find the description of the boundary?

A. You have several inequalities and equalities defining your domain. Choose one inequality and make it an equality, and that is a part of the boundary.
You do this for each inequality and you get different boundary parts.

Describing the boundary of a domain

A compact domain is defined using various equalities and inequalities. Each equality / inequality is called a constraint. (A constraint has only one symbol of equality or inequality.)

Example Domain $\{0 \le x \le 2, 0 \le y \le 3\}$

⇝ Inequality constraints
- ① $0 \le x$
- ② $x \le 2$
- ③ $0 \le y$
- ④ $y \le 3$

Domain $\{x^2+y^2=1, x \ge 0, y \ge 0\}$

⇝ Inequality constraints
- ① $x \ge 0$
- ② $y \ge 0$

Equality constraint: ③ $x^2+y^2=1$

If there are n many inequality constraints, the boundary of the domain has n parts.

Boundary ① : Replace the inequality symbol in inequality constraint ① into the equality.

Boundary ② : Replace the inequality symbol in inequality constraint ② into the equality.

⋮

Boundary ⓝ : Replace the inequality symbol in inequality constraint ⓝ into the equality.

<u>Example</u> Find the boundary of $\{x^2+y^2 \le 1\}$

Inequality constraint : ① $x^2+y^2 \le 1$.

So there is a single boundary part,

Boundary ① : $x^2+y^2=1$.

<u>Example</u> Find the boundary of $\{x^2+y^2 \le 1,\ x \ge 0\}$

Original domain

① $x^2+y^2 \le 1$ } Ineq.
② $x \ge 0$

Change ① →

Boundary ①

① $x^2+y^2=1$ ← Altered

② $x \ge 0$ — Ineq.

Change ② →

Boundary ②

① $x^2+y^2 \le 1$ — Ineq.

② $x=0$ ← Altered

Example Find the boundary of $\{x^2+y^2+z^2=9,\ x+y+z\le 3,\ x\ge 0, y\ge 0, z\ge 0\}$

Original domain

① $x+y+z\le 3$
② $x\ge 0$
③ $y\ge 0$
④ $z\ge 0$
(①–④: Ineq.)
⑤ $x^2+y^2+z^2=9$ – Eq.

Change ① →

Boundary ①

① $x+y+z=3$ ← Altered
② $x\ge 0$
③ $y\ge 0$
④ $z\ge 0$
(②–④: Ineq.)
⑤ $x^2+y^2+z^2=9$ – Eq.

Change ② →

Boundary ②

① $x+y+z\le 3$ – Ineq.
② $x=0$ ← Altered
③ $y\ge 0$
④ $z\ge 0$
(③–④: Ineq.)
⑤ $x^2+y^2+z^2=9$ – Eq.

Change ③ →

Boundary ③

① $x+y+z\le 3$
② $x\ge 0$
(①–②: Ineq.)
③ $y=0$ ← Altered
④ $z\ge 0$ – Ineq.
⑤ $x^2+y^2+z^2=9$ – Eq.

Change ④ ↓

Boundary ④

① $x+y+z\le 3$
② $x\ge 0$
③ $y\ge 0$
(①–③: Ineq.)
④ $z=0$ ← Altered
⑤ $x^2+y^2+z^2=9$ – Eq.

Remark 1. A boundary part may be empty.
2. For each boundary part, some constraints may become redundant.

The method of Lagrange multipliers with multiple equality constraints.

What we learned last time:

Want: Global max/min of $f(x,y,z)$

on domain with <u>one equality constraint</u> $g(x,y,z)=k$

(+ possibly a bunch of inequality constraints)

Lagrange critical point: when $\nabla f(x,y,z)$ and $\nabla g(x,y,z)$ are parallel.

Namely, when $\nabla f(x,y,z)$ and $\nabla g(x,y,z)$ lie on a <u>line</u>.

Concretely, <u>EITHER</u> ① $\nabla g(x,y,z) = \langle 0,0,0 \rangle$

<u>OR</u> ② $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ for some λ.

Two equality constraints

Want: Global max/min of $f(x,y,z)$ on the domain

with <u>2 equality constraints</u> $g(x,y,z)=k$ and $h(x,y,z)=\ell$

(+ possibly a bunch of inequality constraints)

Lagrange critical point:

when $\nabla f(x,y,z)$, $\nabla g(x,y,z)$, $\nabla h(x,y,z)$ lie on a <u>plane</u>.

Concretely,

<u>EITHER</u> ① $\nabla g(x,y,z) = \langle 0,0,0 \rangle$

<u>OR</u> ② $\nabla h(x,y,z) = \langle 0,0,0 \rangle$

<u>OR</u> ③ $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$ for some λ, μ.

Three equality constraints

Want: Global max/min of $f(x,y,z)$ on the domain with <u>3 equality constraints</u> $g(x,y,z)=k$ and
$h(x,y,z)=\ell$ and
$i(x,y,z)=m$

(+ possibly a bunch of inequality constraints)

Lagrange critical points:

<u>EITHER</u> ① $\nabla g(x,y,z) = \langle 0,0,0 \rangle$

<u>OR</u> ② $\nabla h(x,y,z) = \langle 0,0,0 \rangle$

<u>OR</u> ③ $\nabla i(x,y,z) = \langle 0,0,0 \rangle$

<u>OR</u> ④ $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) + \nu \nabla i(x,y,z)$ for some λ, μ, ν.

The pattern continues.

(The conditions for the Lagrange critical points are the same for any number of variables)

Let's do examples.

$\boxed{\text{Example}}$ Find the global max/min values of

$$f(x,y,z) = xy + xz \qquad \text{under 2 constraints,}$$

$$x^2+y^2+z^2=9, \quad 4x+y+z \le 12$$

$\boxed{\text{Solution}}$ There is one equality constraint & one inequality constraint.

$$g(x,y,z)=9 \qquad g(x,y,z) = x^2+y^2+z^2$$

$$h(x,y,z) \le 12 \qquad h(x,y,z) = 4x+y+z$$

$$\nabla f(x,y,z) = \langle y+z, x, x \rangle$$

$$\nabla g(x,y,z) = \langle 2x, 2y, 2z \rangle \quad \nabla h(x,y,z) = \langle 4, 1, 1 \rangle$$

Thus we need to look for the Lagrange critical points with one equality constraint.

$\boxed{\text{Step 1}}$ Find the Lagrange critical points (w/ one equality constraint)

EITHER ① $\nabla g(x,y,z) = \langle 0,0,0 \rangle$

This means $x=y=z=0$, which does not satisfy the equality constraint $x^2+y^2+z^2=9$, so no Lagrange critical points.

OR ② $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

The vector equation is

$$\langle y+z,\ x,\ x\rangle = \lambda\langle 2x, 2y, 2z\rangle = \langle 2\lambda x, 2\lambda y, 2\lambda z\rangle$$

We have $3+1=4$ equations to solve

+ 1 inequality

$\left[\ \boxed{A}\ y+z=2\lambda x \quad \boxed{B}\ x=2\lambda y \quad \boxed{C}\ x=2\lambda z\ \right]$ - from Lagrange

$\boxed{D}\ x^2+y^2+z^2=9$

$\boxed{E}\ 4x+y+z\le 12$

from domain

Compare $\boxed{B}$, $\boxed{C}$

$\boxed{2\lambda y = 2\lambda z}$

$\boxed{2\lambda y = 2\lambda z} \to \boxed{\lambda=0} \xrightarrow[\boxed{A}]{\text{Plug into}} \boxed{y+z=0} \to \boxed{y=-z} \xrightarrow{\boxed{B}} \boxed{x=0} \xrightarrow[\boxed{D}]{\text{Plug into}} \boxed{y^2+(-y)^2=9}$ $\boxed{E}$ ✓

$\boxed{y^2=\frac{9}{2}} \to \boxed{y=\frac{3}{\sqrt2}} \to \boxed{\left(0, \frac{3}{\sqrt2}, -\frac{3}{\sqrt2}\right)}$

$\boxed{y^2=\frac{9}{2}} \to \boxed{y=-\frac{3}{\sqrt2}} \to \boxed{\left(0, -\frac{3}{\sqrt2}, \frac{3}{\sqrt2}\right)}$ $\boxed{E}$ ✓

$\boxed{2\lambda y = 2\lambda z} \to \boxed{y=z} \xrightarrow[\boxed{A}]{\text{Plug into}} \boxed{2y=2\lambda x}$

Plug into $\boxed{B}$

$\boxed{x=2\lambda y=\lambda\cdot(2\lambda x)=2\lambda^2 x}$

$\boxed{x=0} \to \boxed{y=z=0} \to \boxed{(0,0,0)}$ $\boxed{E}$ ✗

$\boxed{2\lambda^2=1} \to \boxed{\lambda=\frac{1}{\sqrt2}} \to \boxed{y=\frac{x}{\sqrt2}=z}$

$\boxed{2\lambda^2=1} \to \boxed{\lambda=-\frac{1}{\sqrt2}} \to \boxed{y=-\frac{x}{\sqrt2}=z}$

PLUG INTO $\boxed{D}$

$\boxed{y=\frac{x}{\sqrt2}=z} \to \boxed{x^2+\frac{x^2}{2}+\frac{x^2}{2}=9}$

$\boxed{x=\frac{3}{\sqrt2}} \to \boxed{\left(\frac{3}{\sqrt2}, \frac{3}{2}, \frac{3}{2}\right)}$ $\boxed{E}$ ✓

$\boxed{x=-\frac{3}{\sqrt2}} \to \boxed{\left(-\frac{3}{\sqrt2}, -\frac{3}{2}, -\frac{3}{2}\right)}$ $\boxed{E}$ ✓

$\boxed{y=-\frac{x}{\sqrt2}=z} \to \boxed{x^2+\frac{x^2}{2}+\frac{x^2}{2}=9}$

$\boxed{x=\frac{3}{\sqrt2}} \to \boxed{\left(\frac{3}{\sqrt2}, -\frac{3}{2}, -\frac{3}{2}\right)}$ $\boxed{E}$ ✓

$\boxed{x=-\frac{3}{\sqrt2}} \to \boxed{\left(-\frac{3}{\sqrt2}, \frac{3}{2}, \frac{3}{2}\right)}$ $\boxed{E}$ ✓

There are 6 Lagrange critical points.

$\left(0, \frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right), \left(0, -\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right), \left(\frac{3}{\sqrt{2}}, \frac{3}{2}, \frac{3}{2}\right), \left(\frac{3}{\sqrt{2}}, -\frac{3}{2}, -\frac{3}{2}\right), \left(-\frac{3}{\sqrt{2}}, \frac{3}{2}, \frac{3}{2}\right), \left(-\frac{3}{\sqrt{2}}, -\frac{3}{2}, -\frac{3}{2}\right)$

$\boxed{\text{Step 2}}$ Find the max/min values of $f(x,y,z)$ at the Lagrange critical points.

$f\left(0, \frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = 0$, $f\left(0, -\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = 0$, $f\left(\frac{3}{\sqrt{2}}, \frac{3}{2}, \frac{3}{2}\right) = \frac{9}{\sqrt{2}}$,

$f\left(\frac{3}{\sqrt{2}}, -\frac{3}{2}, -\frac{3}{2}\right) = -\frac{9}{\sqrt{2}}$, $f\left(-\frac{3}{\sqrt{2}}, \frac{3}{2}, \frac{3}{2}\right) = -\frac{9}{\sqrt{2}}$, $f\left(-\frac{3}{\sqrt{2}}, -\frac{3}{2}, -\frac{3}{2}\right) = \frac{9}{\sqrt{2}}$.

$\Rightarrow$ Max: $\frac{9}{\sqrt{2}}$, Min: $-\frac{9}{\sqrt{2}}$.

$\boxed{\text{Step 3}}$ Find the max/min values of $f(x,y,z)$ on the boundary.

The boundary is described as $\{(x,y,z) \mid x^2+y^2+z^2=9,\ 4x+y+z=12\}$.

For the 4-step process for Step 3, we look for Lagrange critical points with two equality constraints.

$\boxed{\text{Step 3-1}}$ Find the Lagrange critical points (two equality constraints)

EITHER ① $\nabla g(x,y,z) = \langle 0,0,0 \rangle$

This means $\langle 2x, 2y, 2z \rangle = \langle 0,0,0 \rangle$, so $x=y=z=0$. This is impossible due to the equality constraints $x^2+y^2+z^2=9$, $4x+y+z=12$

OR ② $\nabla h(x,y,z) = \langle 0,0,0 \rangle$

This is simply impossible. $\langle 1,1,1 \rangle \neq \langle 0,0,0 \rangle$.

OR ③ $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$ for some λ, μ.

This equation is

$$\langle y+z, x, x\rangle = \lambda\langle 2x, 2y, 2z\rangle + \mu\langle 4,1,1\rangle$$
$$= \langle 2\lambda x+4\mu,\ 2\lambda y+\mu,\ 2\lambda z+\mu\rangle.$$

We have $3+2=5$ equations to solve.

$\boxed{A}$ $y+z = 2\lambda x + 4\mu$
$\boxed{B}$ $x = 2\lambda y + \mu$
$\boxed{C}$ $x = 2\lambda z + \mu$
— from Lagrange multipliers

$\boxed{D}$ $x^2+y^2+z^2 = 9$
$\boxed{E}$ $4x+y+z = 12$
— from equality constraints

Compute $\boxed{B}-\boxed{C}$

$\boxed{B}-\boxed{C}$ $\quad 0 = 2\lambda y - 2\lambda z = 2\lambda(y-z)$

$\boxed{y=z}$ Plug into $\boxed{D}, \boxed{E}$

$\boxed{D}$ $x^2+y^2+y^2 = 9$
$\boxed{E}$ $x+y+y =$

Case 1

$\boxed{\lambda=0}$ Plug into $\boxed{B}, \boxed{C}$

$\boxed{x=\mu}$ Plug into $\boxed{A}$

$\boxed{y+z=4x}$ Plug into $\boxed{D}, \boxed{E}$

$\boxed{D}$ $\left(\frac{y+z}{4}\right)^2 + y^2+z^2 = 9$
$\boxed{E}$ $(y+z)+y+z = 12$

Case 2

[Here we immediately jumped into $\boxed{D}, \boxed{E}$ as we don't care so much about knowing what λ, μ are.]

Case 1:

$\boxed{D}$ $x^2+2y^2=9$

$\boxed{E}$ $4x+2y=12$

$\xrightarrow{\text{Use } \boxed{E}}$ $x=\frac{12-2y}{4}=\frac{6-y}{2}=3-\frac{y}{2}$

Plug into $\boxed{D}$

$\left(3-\frac{y}{2}\right)^2+2y^2=9 \longrightarrow \frac{9}{4}y^2-3y+9=9$

$\frac{9}{4}y^2-3y=0 \longrightarrow 3y\left(\frac{3y}{4}-1\right)=0$

$x=3-\frac{y}{2}, \quad z=y$

$y=0 \longrightarrow x=3, \ z=0 \longrightarrow (3,0,0)$

$y=\frac{4}{3} \longrightarrow x=\frac{7}{3}, \ z=\frac{4}{3} \longrightarrow \left(\frac{7}{3},\frac{4}{3},\frac{4}{3}\right)$

Case 2:

$\boxed{D}$ $\left(\frac{y+z}{4}\right)^2+y^2+z^2=9$

$\boxed{E}$ $2(y+z)=12$

$\xrightarrow{\text{Use } \boxed{E}}$ $y=6-z$

Plug into $\boxed{D}$

$\left(\frac{6}{4}\right)^2+(6-z)^2+z^2=9 \longrightarrow 2z^2-12z+\frac{117}{4}=0$

$2(z-3)^2+\frac{45}{4}=0 \longrightarrow$ No solution!

Thus, the Lagrange critical points are $(3,0,0), (\frac{7}{3}, \frac{4}{3}, \frac{4}{3})$.

$\boxed{\text{Step 3-2}}$ Find the max/min values of $f(x,y,z)$ at the Lagrange critical points.

$f(3,0,0) = 0 \leftarrow$ Min $\quad f(\frac{7}{3}, \frac{4}{3}, \frac{4}{3}) = \frac{7}{3}\cdot\frac{4}{3} + \frac{7}{3}\cdot\frac{4}{3} = \frac{56}{9} \leftarrow$ Max.

$\boxed{\text{Step 3-3}}$ Find the max/min values of $f(x,y,z)$ on the boundary.

The domain only has equality constraints and no inequality constraints $\Rightarrow$ No boundary.

$\boxed{\text{Step 3-4}}$ Compare.

	Lagrange critical	Boundary
Max	$\frac{56}{9}$	X
Min	0	X

$\rightsquigarrow$ Global max: $\frac{56}{9}$, Global min: 0.

$\boxed{\text{Step 4}}$ Compare.

	Lagrange critical	Boundary
Max	$\frac{9}{\sqrt{2}}$	$\frac{56}{9}$
Min	$-\frac{9}{\sqrt{2}}$	0

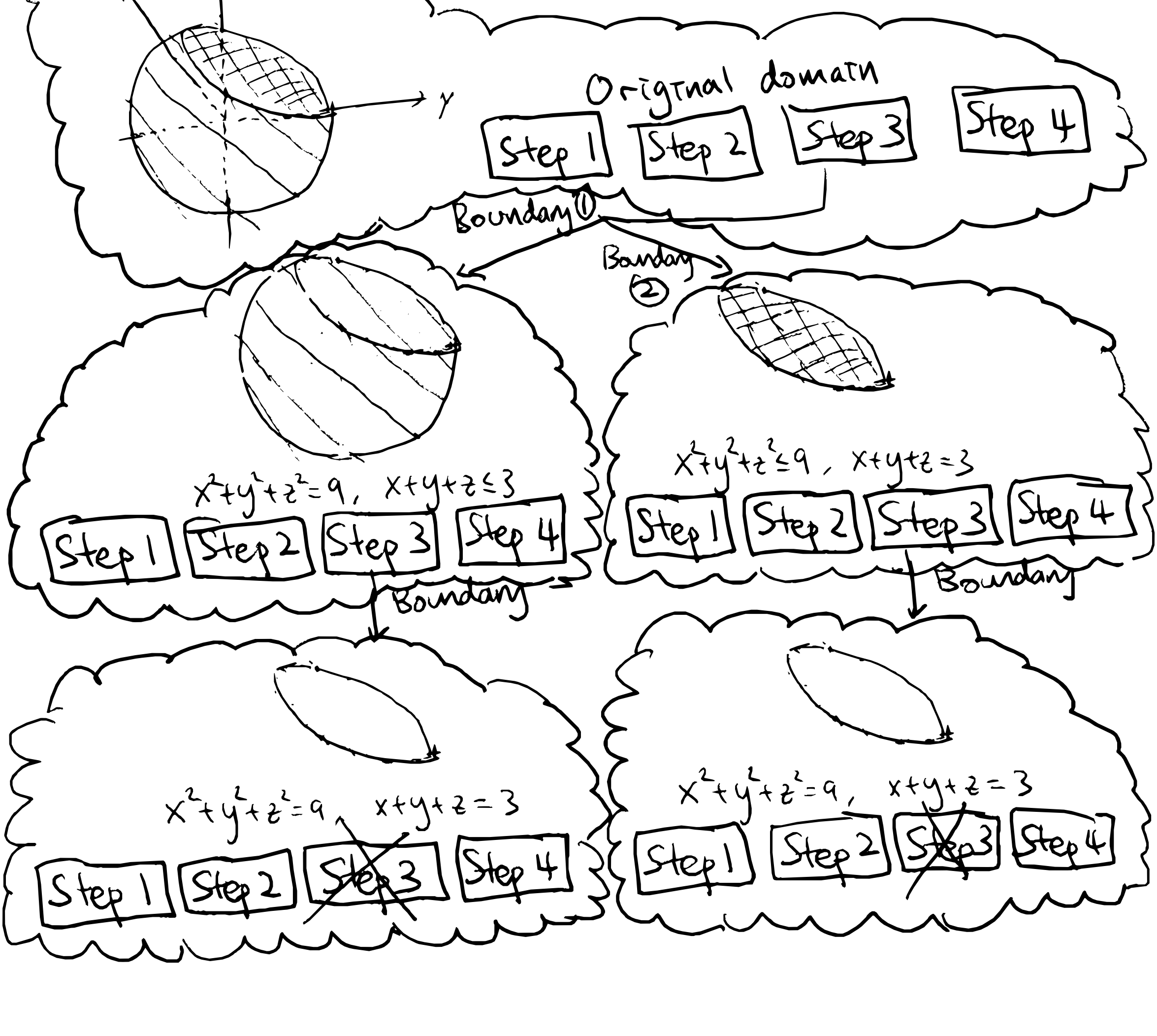
$\rightsquigarrow$ Global max: $\frac{9}{\sqrt{2}}$, Global min: $-\frac{9}{\sqrt{2}}$.

The situation becomes more complicated if there are boundaries (or boundaries of boundaries)

Tip for running the 4-step process more efficiently

Say the domain is $x^2+y^2+z^2 \le 9$, $x+y+z \le 3$.

Imagine running the 4-step process with this domain.



If you honestly follow the recursion of the 4-step processes as before, there are a lot of redundancies.

- The boundary of Boundary ① is the same as the boundary of Boundary ②, so $\boxed{\text{Step 3}}$ of Boundary ① is identical to $\boxed{\text{Step 3}}$ of Boundary ②.
- $\boxed{\text{Step 2}}$, $\boxed{\text{Step 4}}$ are just bookkeeping steps, but you don't have to bookkeep that often.

Therefore, you can reduce the recursive $\boxed{\text{Step 3}}$ of the original domain into the following:

$\boxed{\text{Step 3}}$
- Identify the boundaries, the boundaries of the boundaries, ... (Make any number of inequality constraints into equality constraints.)
- For each of them, find the Lagrange critical points.
- Collect all the Lagrange critical points you found, and find the max/min values at those points.

This means that basically for sub-problems in $\boxed{\text{Step 3}}$, you only need to do $\boxed{\text{Step 1}}$s.

Example Find the global max/min values of

$$f(x,y,z) = x^3+y^3+z^3$$

on the domain $\{x^2+y^2+z^2 \le 9,\ x+y+z \le 3\}$.

Solution The domain has 2 inequality constraints,

$g(x,y,z) \le 9$ where $g(x,y,z) = x^2+y^2+z^2$,

$h(x,y,z) \le 3$ where $h(x,y,z) = x+y+z$,

and no equality constraints. $\Rightarrow$ Look for critical points in $\boxed{\text{Step 1}}$.

$\boxed{\text{Step 1}}$ Find the critical points.

$\nabla f(x,y,z) = \langle 3x^2, 3y^2, 3z^2\rangle$, so $\nabla f(x,y,z) = \langle 0,0,0\rangle$

means $x=y=z=0$ $\rightsquigarrow$ critical point $(0,0,0)$.

$\boxed{\text{Step 2}}$ Find the max/min values of $f(x,y,z)$ at the critical points.

$$f(0,0,0) = 0.$$

$\boxed{\text{Step 3}}$ Let's use the more efficient version.

Identify all boundaries, boundaries of boundaries, ...

Perform $\boxed{\text{Step 1}}$s for each, and find the max/min values of $f(x,y,z)$ at the Lagrange critical points.

Finding all boundaries, boundaries of boundaries, . . .
amounts to flipping any number of inequality constraints
to equality constraints.

Original domain

$x^2+y^2+z^2 \le 9$

$x+y+z \le 3$

Flipping 1 inequality ⇒ equality

Boundary

Boundary ①

$x^2+y^2+z^2 = 9$

$x+y+z \le 3$

Boundary ②

$x^2+y^2+z^2 \le 9$

$x+y+z = 3$

Flipping 2 inequalities ⇒ equalities

Boundary of boundary

Boundary of boundary ③

$x^2+y^2+z^2 = 9$

$x+y+z = 3$

Note that we set $g(x,y,z) = x^2+y^2+z^2 \rightsquigarrow \nabla g(x,y,z) = \langle 2x, 2y, 2z \rangle$

$h(x,y,z) = x+y+z \rightsquigarrow \nabla h(x,y,z) = \langle 1,1,1 \rangle$.

$\boxed{\text{Step 1}}$ for $\boxed{\text{Boundary ①}}$ $g(x,y,z) = 9 \leftarrow$ Equality

$h(x,y,z) \le 3 \leftarrow$ Inequality

Lagrange critical points w/ 1 equality constraint.

EITHER ① $\nabla g(x,y,z) = \langle 0,0,0 \rangle$

This means $\langle 2x, 2y, 2z \rangle = \langle 0,0,0 \rangle$, so $x=y=z=0$

but this is inconsistent with the equality constraint

$x^2+y^2+z^2=9$. $\rightsquigarrow$ Disregard.

OR ② $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ for some λ

This means $\langle 3x^2, 3y^2, 3z^2 \rangle = \lambda \langle 2x, 2y, 2z \rangle$

$= \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle$

so we need to solve 3+1 equations + 1 inequality.

Equations:

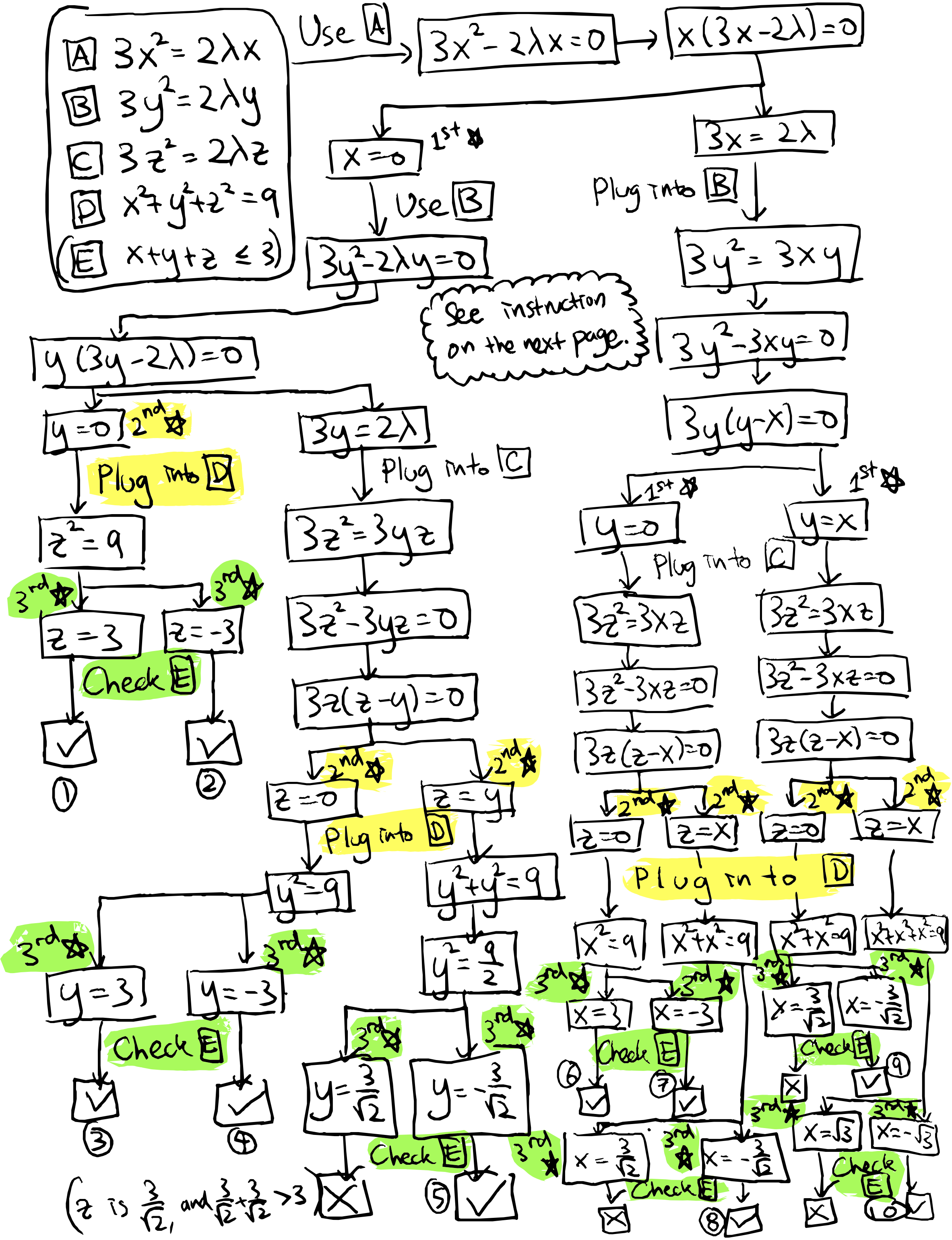
- $\boxed{A}$ $3x^2 = 2\lambda x$
- $\boxed{B}$ $3y^2 = 2\lambda y$
- $\boxed{C}$ $3z^2 = 2\lambda z$

from Lagrange multipliers

- $\boxed{D}$ $x^2+y^2+z^2=9$

Inequality — $\boxed{E}$ $x+y+z \le 3$

from constraints.



A 3x² = 2λx
B 3y² = 2λy
C 3z² = 2λz
D x² + y² + z² = 9
(E x + y + z ≤ 3)
Use A
3x² − 2λx = 0
x(3x − 2λ) = 0
x = 0
1st
3x = 2λ
Use B
Plug into B
3y² − 2λy = 0
3y² = 3xy
See instruction on the next page.
y(3y − 2λ) = 0
3y² − 3xy = 0
3y(y − x) = 0
y = 0
2nd
3y = 2λ
Plug into C
Plug into D
3z² = 3yz
y = 0
1st
y = x
1st
z² = 9
Plug into C
3z² − 3yz = 0
3z² = 3xz
3z² = 3xz
3rd
3rd
z = 3
z = −3
3z(z − y) = 0
3z² − 3xz = 0
3z² − 3xz = 0
Check E
3z(z − x) = 0
3z(z − x) = 0
①
②
2nd
2nd
z = 0
z = y
2nd
2nd
2nd
2nd
Plug into D
z = 0
z = x
z = 0
z = x
y² = 9
y² + y² = 9
Plug into D
3rd
3rd
y² = 9/2
x² = 9
x² + x² = 9
x² + x = 9
x² + x² + x² = 9
y = 3
y = −3
3rd
3rd
3rd
3rd
Check E
3rd
3rd
x = 3
x = −3
x = 3/√2
x = −3/√2
y = 3/√2
y = −3/√2
Check E
Check E
③
④
⑥
⑦
⑨
Check E
3rd
3rd
3rd
3rd
x = 3/√2
x = −3/√2
x = √3
x = −√3
(z is 3/√2, and 3/√2 + 3/√2 > 3)
⑤
Check E
Check E
⑧
⑩

We have 10 Lagrange critical points

	First ☆	Second ☆	Third ☆
① : $(0,0,3)$	$\boxed{x=0}$	$\boxed{y=0}$	$\boxed{z=3}$
② : $(0,0,-3)$	$\boxed{x=0}$	$\boxed{y=0}$	$\boxed{z=-3}$
③ : $(0,3,0)$	$\boxed{x=0}$	$\boxed{z=0}$	$\boxed{y=3}$
④ : $(0,-3,0)$	$\boxed{x=0}$	$\boxed{z=0}$	$\boxed{y=-3}$
⑤ : $(0,-\frac{3}{\sqrt{2}},-\frac{3}{\sqrt{2}})$	$\boxed{x=0}$	$\boxed{z=y}$	$\boxed{y=-\frac{3}{\sqrt{2}}}$
⑥ : $(3,0,0)$	$\boxed{y=0}$	$\boxed{z=0}$	$\boxed{x=3}$
⑦ : $(-3,0,0)$	$\boxed{y=0}$	$\boxed{z=0}$	$\boxed{x=-3}$
⑧ : $(-\frac{3}{\sqrt{2}},0,-\frac{3}{\sqrt{2}})$	$\boxed{y=0}$	$\boxed{z=x}$	$\boxed{x=-\frac{3}{\sqrt{2}}}$
⑨ : $(-\frac{3}{\sqrt{2}},-\frac{3}{\sqrt{2}},0)$	$\boxed{y=x}$	$\boxed{z=0}$	$\boxed{x=-\frac{3}{\sqrt{2}}}$
⑩ : $(-\sqrt{3},-\sqrt{3},-\sqrt{3})$	$\boxed{y=x}$	$\boxed{z=x}$	$\boxed{x=-\sqrt{3}}$

In the previous flow chart..

- ☆ denotes an equation that determines x,y,z (not involving λ)
- Our goal is to determine the possible solutions to x,y,z, not to remember the values of λ. Thus, we look for three ☆s on each possible case. (because there are three unknowns, x,y,z)
- As the equality constraint (equation $\boxed{D}$) gives one equation only involving x,y,z, for each possible case, we moved on to plugging into $\boxed{D}$ as soon as we had two ☆s.

- After solving for the equations (namely, after facing the third ☆), the last thing to check is to check if the point you found satisfies the inequality constraint(s).

$\boxed{\text{Step 1}}$ for $\boxed{\text{Boundary ②}}$

$g(x,y,z) \le 9$ ← Inequality
$h(x,y,z) = 3$ ← Equality

Lagrange critical points w/ 1 equality constraint.

EITHER ① $\nabla h(x,y,z) = \langle 0,0,0 \rangle$

Since $\nabla h(x,y,z) = \langle 1,1,1 \rangle$, this is impossible.

OR ② $\nabla f(x,y,z) = \lambda \nabla h(x,y,z)$ for some λ

This means $\langle 3x^2, 3y^2, 3z^2 \rangle = \lambda \langle 1,1,1 \rangle = \langle \lambda, \lambda, \lambda \rangle$

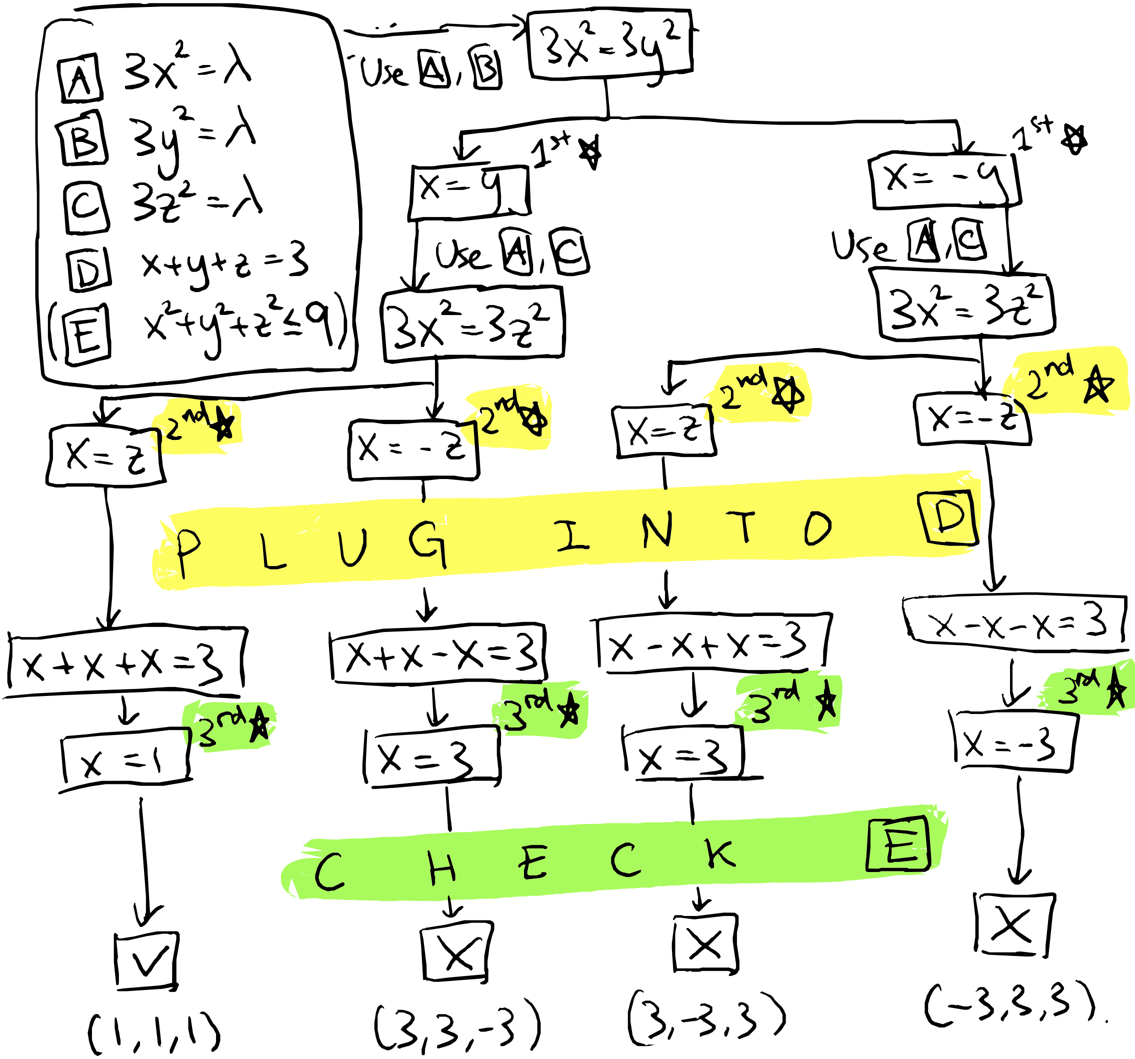
so we need to solve 3+1 equations + 1 inequality

Equations —
$\boxed{A}$ $3x^2 = \lambda$
$\boxed{B}$ $3y^2 = \lambda$
$\boxed{C}$ $3z^2 = \lambda$
from Lagrange multipliers

$\boxed{D}$ $x+y+z = 3$

Inequality — $\boxed{E}$ $x^2+y^2+z^2 \le 9$

from constraints.



There is one Lagrange critical point, $(1,1,1)$.

$\boxed{\text{Step 1}}$ for $\boxed{\text{Boundary of boundary ③}}$

$g(x,y,z) = 9$
$h(x,y,z) = 3$
} Equalities

$\Rightarrow$ Lagrange critical points w/ 2 equality constraints.

EITHER ① $\nabla g(x,y,z) = \langle 0,0,0 \rangle$

This means $\langle 2x, 2y, 2z \rangle = \langle 0,0,0 \rangle$, so $x=y=z=0$, which violates the equality constraints, $x^2+y^2+z^2=9$ and $x+y+z=3$.

$\rightsquigarrow$ Disregard.

OR ② $\nabla h(x,y,z) = \langle 0,0,0 \rangle$

This is impossible since $\nabla h(x,y,z) = \langle 1,1,1 \rangle$.

OR ③ $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$

This equation is

$$\langle 3x^2, 3y^2, 3z^2 \rangle = \lambda \langle 2x, 2y, 2z \rangle + \mu \langle 1,1,1 \rangle$$
$$= \langle 2\lambda x + \mu, 2\lambda y + \mu, 2\lambda z + \mu \rangle.$$

So, we need to solve 3+2 equations.

Equations

$\boxed{A}$ $3x^2 = 2\lambda x + \mu$
$\boxed{B}$ $3y^2 = 2\lambda y + \mu$ — from Lagrange multipliers
$\boxed{C}$ $3z^2 = 2\lambda z + \mu$
$\boxed{D}$ $x^2+y^2+z^2=9$ — from constraints.
$\boxed{E}$ $x+y+z=3$

We'll draw the flowchart on the next page. Note that since we have 2 equality constraints and we look for 3 unknowns (x,y,z), we will move on to plugging into $\boxed{D}$ and $\boxed{E}$ after encountering the 1st ☆.

A $3x^2 = 2\lambda x + \mu$

B $3y^2 = 2\lambda y + \mu$

C $3z^2 = 2\lambda z + \mu$

D $x^2 + y^2 + z^2 = 9$

E $x + y + z = 3$

A − B

$$3(x^2 - y^2) = 2\lambda(x - y)$$

$$3(x+y)(x-y) = 2\lambda(x-y)$$

$$(3x + 3y - 2\lambda)(x - y) = 0$$

$x = y$ 1st ★

Plug into D, E

D $2x^2 + z^2 = 9$

E $2x + z = 3$

2nd ★

$z = 3 - 2x$, and

$2x^2 + (3 - 2x)^2 = 9$

$$6x^2 - 12x + 9 = 9$$

$$6x^2 - 12x = 0$$

$$6x(x - 2) = 0$$

$x = 0$ 3rd ★ (1)

$x = 2$ 3rd ★ (2)

$2\lambda = 3x + 3y$

Plug into B

$$3y^2 = (3x + 3y)y + \mu$$

$$3y^2 = 3xy + 3y^2 + \mu$$

$$\mu = -3xy$$

Plug into C

$$3z^2 = (3x + 3y)z - 3xy$$

$$3z^2 = 3xz + 3yz - 3xy$$

$$z^2 - xz - yz + xy = 0$$

$$(z - x)(z - y) = 0$$

$z = x$ 1st ★

$z = y$ 1st ★

PLUG INTO D, E

D $2x^2 + y^2 = 9$

E $2x + y = 3$

2nd ★

$y = 3 - 2x$, and

$2x^2 + (3 - 2x)^2 = 9$

$$6x^2 - 12x + 9 = 9$$

$$6x^2 - 12x = 0$$

$$6x(x - 2) = 0$$

$x = 0$ 3rd ★ (3)

$x = 2$ 3rd ★ (4)

D $x^2 + 2y^2 = 9$

E $x + 2y = 3$

2nd ★

$x = 3 - 2y$, and

$(3 - 2y)^2 + 2y^2 = 9$

$$6y^2 - 12y + 9 = 9$$

$$6y^2 - 12y = 0$$

$$6y(y - 2) = 0$$

(5) 3rd ★ $y = 0$

(6) 3rd ★ $y = 2$

There are 6 Lagrange critical points.

		1st ☆	2nd ☆	3rd ☆
①	$(0,0,3)$	$\boxed{x=y}$	$\boxed{z=3-2x}$	$\boxed{x=0}$
②	$(2,2,-1)$	$\boxed{x=y}$	$\boxed{z=3-2x}$	$\boxed{x=2}$
③	$(0,3,0)$	$\boxed{z=x}$	$\boxed{y=3-2x}$	$\boxed{x=0}$
④	$(2,-1,2)$	$\boxed{z=x}$	$\boxed{y=3-2x}$	$\boxed{x=2}$
⑤	$(3,0,0)$	$\boxed{z=y}$	$\boxed{x=3-2y}$	$\boxed{y=0}$
⑥	$(-1,2,2)$.	$\boxed{z=y}$	$\boxed{x=3-2y}$	$\boxed{y=2}$

(As there are no inequality constraints, there were no extra checking after 3rd ☆.)

So.. now we have everything.

Boundary ①

$f(0,0,3) = 27$ | $f(3,0,0) = 27$
$f(0,0,-3) = -27$ | $f(-3,0,0) = -27$
$f(0,3,0) = 27$ | $f\left(-\frac{3}{\sqrt{2}},0,-\frac{3}{\sqrt{2}}\right) = -\frac{27}{\sqrt{2}}$
$f(0,-3,0) = -27$ | $f\left(-\frac{3}{\sqrt{2}},-\frac{3}{\sqrt{2}},0\right) = -\frac{27}{\sqrt{2}}$
$f\left(0,-\frac{3}{\sqrt{2}},-\frac{3}{\sqrt{2}}\right) = -\frac{27}{\sqrt{2}}$ | $f(-\sqrt{3},-\sqrt{3},-\sqrt{3}) = -9\sqrt{3}$

Boundary ②

$f(1,1,1) = 3$

Boundary of boundary ③

$f(0,0,3) = 27$ | $f(2,-1,2) = 15$
$f(2,2,-1) = 15$ | $f(3,0,0) = 27$
$f(0,3,0) = 27$ | $f(-1,2,2) = 15$

So.. out of these values.

Max : 27, Min : -27.

$\boxed{\text{Step 4}}$ Compare.

	Critical	Boundary
Max	0	27
Min	0	-27

⇒ Global max : 27

Global min : -27.